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On the out-of-equilibrium order parameter in long-range spin-glasses

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Abstract. We show that the dynamical order parameters can be re-expressed in terms of the distribution of the staggered auto-correlation and response functions. We calculate these distributions for the out of equilibrium dynamics of the Sherrington–Kirkpatrick model at long times. The results suggest that the landscape this model visits at different long times in an out-of-equilibrium relaxation process is, in a sense, self-similar. Furthermore, there is a similarity between the landscape seen out of equilibrium at long times and the equilibrium landscape.

The calculation is greatly simplified by making use of the superspace notation in the dynamical approach. This notation also highlights the rather mysterious formal connection between the dynamical and replica approaches.

We also perform numerical simulations which show good agreement with the analytical results for the out of equilibrium dynamics.

1. Introduction

The partition function of mean-field spin-glasses is dominated by many states. The geometrical organization of these states, their relative weights in the Gibbs–Boltzmann measure, and the distribution of their mutual distances have been known for some time [1, 2]. Of particular importance is the functional order parameter $P(q)$ giving the probability distribution of states with mutual overlap q.

The Gibbs–Boltzmann measure can be studied analytically using a dynamical approach [3]. For instance, the Langevin dynamics

$$\Gamma_0^{-1}\,\partial_t\sigma_i(t) = -\beta\frac{\delta H}{\delta\sigma_i(t)} + \xi_i(t) \tag{1.1}$$

(Γ_0 determines the time scale and $\xi_i(t)$ is a Gaussian white noise with zero mean and variance 2), with the following order of large times and large N limits,

$$\lim_{N\to\infty}\ \lim_{t\to\infty} \tag{1.2}$$

guarantees ergodicity and leads the system to equilibrium. The equilibrium thermodynamical values of any operator O are then obtained as averages over the noise $\langle O\rangle_{\rm eq} = \lim_{N\to\infty}\lim_{t\to\infty}\langle O(t)\rangle$.

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A different situation, closer to the experimental settings, is to consider the relaxation of an infinite system at long but finite times. The time is measured from the initial time—the quenching time in experiments—which we take as zero. The order of limits is then

$$\lim_{t\to\infty} \lim_{N\to\infty} . \tag{1.3}$$

An analytical solution for mean-field spin-glasses in this regime has recently been developed [4–7]. It was argued there that in the regime (1.3) mean-field spin-glass systems *below* the critical temperature never achieve equilibrium, not even within a restricted sector of phase space. This is in agreement with experimental spin-glasses, for which the estimate is that aging effects take a few years to die away [8].

A relevant order parameter for the long-time asymptotics of the relaxation is the dynamical $P_{\mathrm{d}}(q)$ defined as follows [4]: we add time-independent source terms $h_{i_1\dots i_r}$ to the energy

$$H_h = H + \frac{1}{N^{r-1}} \sum_{i_1,\dots,i_r}^{N} h_{i_1\dots i_r}\sigma_{i_1}\dots\sigma_{i_r} \tag{1.4}$$

and then consider the generating functions of the generalized susceptibilities

$$\mathrm{Lim}\left[1 - \frac{r}{N^r}\sum_{i_1<\dots<i_r} \left.\frac{\partial\overline{\langle s_{i_1}(t)\cdots s_{i_r}(t)\rangle}}{\partial h_{i_1\dots i_r}}\right|_{h=0}\right]$$
$$= \int_0^1 \mathrm{d}q'\,\frac{\mathrm{d}X_{()}(q')}{\mathrm{d}q'}\,q'^r = \int_0^1 \mathrm{d}q'\,P_{()}(q')\,q'^r . \tag{1.5}$$

If the symbol Lim stands for (1.2), it defines the usual Parisi order parameters $x(q)$ and $P(q)$ [1]. If it stands for (1.3) then (1.5) defines the dynamical parameters $X_{\mathrm{d}}(q)$ and $P_{\mathrm{d}}(q)$ [4]. Here and in what follows $\langle\cdot\rangle$ and overline denote the average with respect to the Langevin noise and with respect to the couplings, respectively.

The dynamical order parameters $X_{\mathrm{d}}(q)$ and $P_{\mathrm{d}}(q)$—unlike their static counterparts—have not as yet been given a probabilistic interpretation. The main purpose of this paper is to show that $P_{\mathrm{d}}(q)$ can be recast into a form that:

(i) Makes its physical meaning more explicit.
(ii) Shows for the Sherrington–Kirkpatrick (SK) model that there is a self-similarity in the landscape. Although at all long but finite times (limit (1.3)) the system is exploring regions of phase space which it will eventually leave, never to return, some geometrical properties of these regions coincide with those of the equilibrium states.
(iii) Is amenable to numerical simulations.

The Hamiltonian of the Sherrington–Kirkpatrick model is given by

$$H(\sigma) = \frac{1}{N}\sum_{ij} J_{ij}\sigma_i\sigma_j + a\sum_i \left(\sigma_i^2 - 1\right)^2 . \tag{1.6}$$

The interactions J_{ij} are quenched random variables Gaussianly distributed with zero mean and variance $1/\sqrt{N}$. The last is a spin weight term and the hard-spin limit (± 1) is recovered taking $a \to \infty$. In what follows we shall work in this limit.

The out-of-equilibrium dynamics of the SK model have been studied in [6]. A surprising outcome, obtained under a set of hypotheses described there in detail, is that the dynamical order parameters $P_{\mathrm{d}}(q)$ and $X_{\mathrm{d}}(q)$ coincide with the static order parameters $P(q)$ and $X(q)$—even if the physical situations they describe are very different. The coincidence of dynamical and static order parameters does not hold for every model, for instance the

p-spin spherical model [9] behaves in a different way [4]. This difference in behaviour can be understood by studying the TAP landscapes of the two models [4].

We shall use in this paper the results of [6] to compute analytically the staggered auto-correlation and response functions [10, 11] in the limit of any two large-times (limit (1.3)) for the SK model. All the information about the asymptotic large times solution is encoded in the order parameter $X_d(q)$ and in the 'triangle relation' f relating the correlation functions at any three long times (see [6]). We shall then compare analytical and numerical results for the staggered distributions. Their good agreement gives numerical support for the predicted equality of dynamical and static order parameters in this case.

In order to obtain the staggered distributions from $X_d(q)$ (or $P_d(q)$) and f we shall heavily use the *formal* relation between the static replica approach and the dynamical approach, which becomes transparent when the latter is formulated in terms of superspace variables [12, 13]. (Although the underlying supersymmetry in this dynamics is partially broken by the 'boundary'—initial—conditions, it still has useful consequences.)

This proceeds in two steps: we firstly identify the dynamical—superspace—counterparts of the static—replica space—variables. We obtain, roughly speaking, the same formulæ with superspace integrals—including a time integral—substituting sums over replicas.

Secondly, we look for the solution for the dynamical order parameter. Again, this solution has many points in common with the solution for the statics although they are not equivalent for every model and describe entirely different physical situations.

The paper is organised as follows. In section 2 we introduce the staggered distributions. Using the SUSY formalism, we find that they are related to the powers of the dynamical order parameter $Q(1, 2)$. In section 3 we compute, in general, these powers of $Q(1, 2)$ and then specialize to the long-time asymptotics of the SK model using the results of [6]. In section 4 we obtain the staggered auto-correlation function for large times. In section 5 we describe the numerical simulations and compare them to the analytical results. Finally, we discuss the physical picture in section 6.

2. Staggered distribution functions

The staggered auto-correlation function is defined as

$$g(\lambda; t_1, t_2) \equiv \langle \sigma_\lambda(t_1)\, \sigma_\lambda(t_2) \rangle = \sum_{ij} \langle \lambda | i \rangle \, \langle \lambda | j \rangle \, \langle \sigma_i(t_1)\, \sigma_j(t_2) \rangle \tag{2.1}$$

where λ denotes the eigenvalues of the $N \times N$ random matrix J_{ij} associated with the eigenvectors $|\lambda\rangle$. $|\sigma(t)\rangle$ is the time-dependent N-dimensional vector of spins, $\sigma_i(t) \equiv \langle i | \sigma(t) \rangle$, and $\sigma_\lambda(t) \equiv \langle \lambda | \sigma(t) \rangle$ are the staggered spin states.

The staggered response function is

$$\hat{g}(\lambda; t_1, t_2) \equiv \left\langle \frac{\delta \sigma_\lambda(t_1)}{\delta h_\lambda(t_2)} \right\rangle = \sum_{ij} \langle \lambda | i \rangle \, \langle \lambda | j \rangle \left\langle \frac{\delta \sigma_i(t_1)}{\delta h_j(t_2)} \right\rangle . \tag{2.2}$$

The functions g and $\hat{g}$ are in turn related to the set of time-dependent two-point functions

$$E^{(k)}(t_1, t_2) \equiv \frac{1}{N} \sum_{ij} \overline{\left\langle \sigma_i(t_1) \, \left(J^k\right)_{ij} \, \sigma_j(t_2) \right\rangle} \tag{2.3}$$

$$\hat{E}^{(k)}(t_1, t_2) \equiv \frac{1}{N} \sum_{ij} \overline{\left\langle \left(J^k\right)_{ij} \, \frac{\delta \sigma_i(t_1)}{\delta h_j(t_2)} \right\rangle} \tag{2.4}$$

where $\overline{\cdot}$ represents the mean over the quenched disorder. In terms of g and $\hat{g}$ they read

$$E^{(k)}(t_1, t_2) \equiv \int \mathrm{d}\lambda\, \rho(\lambda)\, \lambda^k\, g(\lambda; t_1, t_2) \tag{2.5}$$

$$\hat{E}^{(k)}(t_1, t_2) \equiv \int \mathrm{d}\lambda\, \rho(\lambda)\, \lambda^k\, \hat{g}(\lambda; t_1, t_2) \tag{2.6}$$

where $\rho(\lambda)$ is the eigenvalue distribution that, in the limit of large N, corresponds to the semicircle law $\rho(\lambda) = 1/(2\pi)\sqrt{4-\lambda^2}$ if the variance of the J_{ij} is finite [14]. In particular, if $t_1 = t_2$ and $k = 1$ equation (2.5) gives the time-dependent energy density.

2.1. Supersymmetric formalism

Following [12] we introduce the supersymmetric 'field' $\phi_i(1)$, $i = 1, \ldots, N$,

$$\phi_i(1) \equiv \sigma_i(t_1) + \eta_i(t_1)\overline{\theta}_1 + \theta_1\overline{\eta}_i(t_1) + \hat{\sigma}_i(t_1)\overline{\theta}_1\theta_1 \tag{2.7}$$

with $1 \equiv (t_1, \theta_1, \overline{\theta}_1)$.

The dynamical expectation value of a quantity O can then be written as

$$\begin{aligned}
\langle O(t_1)\rangle &= \int \Pi_i D[\phi_i]\, O(t_1) \exp[-S_{\mathrm{KIN}} - S_{\mathrm{POT}}] \\
S_{\mathrm{KIN}} &= \Gamma_0^{-1} \int \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, \mathrm{d}t \sum_i \frac{\partial\phi_i}{\partial\theta}\left(\frac{\partial\phi_i}{\partial\bar{\theta}} - \theta\frac{\partial\phi_i}{\partial t}\right) \\
S_{\mathrm{POT}} &= \beta \int \mathrm{d}\theta\, \mathrm{d}\bar{\theta}\, \mathrm{d}t\, H(\phi)\,.
\end{aligned} \tag{2.8}$$

As in the static replica approach, once the mean is taken over the couplings one ends up with a functional of the order parameters that can be calculated by saddle-point evaluation. The dynamical order parameter is the 'supercorrelation' function defined as

$$Q(1,2) \equiv \frac{1}{N}\sum_i \langle\phi_i(1)\phi_i(2)\rangle \tag{2.9}$$

which plays the same role as Q_{ab} in the statics. For the mean-field case that satisfies causality the saddle-point value of $Q(1,2)$ can be written as

$$Q(1,2) = C(t_1,t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\,[\theta_2\; G(t_1,t_2) + \theta_1\; G(t_2,t_1)] \tag{2.10}$$

and it encodes the two-time functions C and G that are the standard auto-correlation and response functions

$$C(t_1,t_2) \equiv \frac{1}{N}\sum_i \langle\sigma_i(t_1)\,\sigma_i(t_2)\rangle \tag{2.11}$$

$$G(t_1,t_2) \equiv \frac{1}{N}\sum_i \left\langle\frac{\delta\sigma_i(t_1)}{\delta h_i(t_2)}\right\rangle \tag{2.12}$$

respectively. Because of causality, $G(t_1,t_2) = 0$ if $t_2 > t_1$. In these formulæ we have omited the mean over the disorder since C and G are self-averaging in the limit $N \to \infty$ *for finite times* as can be easily proven by considering the evolution of two independent copies of the system with the same couplings J_{ij}.

We shall need the definition of the operator powers of Q

$$Q^k(1,3) \equiv \int \mathrm{d}2\, Q^{k-1}(1,2)\, Q(2,3)\,. \tag{2.13}$$

It is easy to see that $\boldsymbol{Q}^k$ conserves the form (2.10) with $C^{(k)}$ and $G^{(k)}$ given inductively by

$$C^{(k)}(t_1, t_3) = \int \mathrm{d}t_2 \left[C^{(k-1)}(t_1, t_2)\, G(t_3, t_2) + G^{(k-1)}(t_1, t_2)\, C(t_2, t_3)\right] \tag{2.14}$$

$$G^{(k)}(t_1, t_3) = \int \mathrm{d}t_2\, G^{(k-1)}(t_1, t_2)\, G(t_2, t_3) \tag{2.15}$$

where $t_1 > t_3$. From now on supra-indices within parenthesis denote entries in the function $\boldsymbol{Q}^k$ while supra-indices without parentheses denote ordinary powers.

2.2. Staggered auto-correlation and response functions

We now start the computation of the staggered auto-correlations and responses. With the superspace notation most of the manipulations of [11] carry through without change, just substituting replica indices by superspace variables. The quantities (2.1) and (2.2) can be encoded in a function of superspace variables

$$\begin{aligned}\boldsymbol{E}^{(k)}(1, 2) &= \frac{1}{N} \sum_{ij} \overline{\langle\langle \phi_i(1) \left(J^k\right)_{ij} \phi_j(2)\rangle\rangle} \\ &= E^{(k)}(t_1, t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\left[\theta_2\, \hat{E}^{(k)}(t_1, t_2) + \theta_1\, \hat{E}^{(k)}(t_2, t_1)\right]\end{aligned} \tag{2.16}$$

where '$\langle\langle\cdot\rangle\rangle$' denotes mean with the measure (2.8).

Correspondingly, the staggered distributions can be encoded as

$$\boldsymbol{g}(\lambda; 1, 2) \equiv g(\lambda; t_1, t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\left[\theta_2\, \hat{g}(\lambda; t_1, t_2) + \theta_1\, \hat{g}(\lambda; t_2, t_1)\right]. \tag{2.17}$$

We shall use a related set of order parameters†

$$\boldsymbol{\mathcal{X}}^{(k)}(1, 2) \equiv \sum_{r=0}^{k} S_{k,r}\, \boldsymbol{E}^{(r)}(1, 2) \tag{2.18}$$

where $S_k(z) = \sum_{r=0}^{k} S_{k,r}\, z^r$ are the Chebyshev polynomials of the second kind, generated by

$$\sum_{k=0}^{\infty} S_k(z)\, y^k = \frac{1}{(1 - yz + y^2)}. \tag{2.19}$$

In components, $\boldsymbol{\mathcal{X}}^{(k)}(1, 2)$ reads

$$\boldsymbol{\mathcal{X}}^{(k)}(1, 2) \equiv \mathcal{X}^{(k)}(t_1, t_2) + (\overline{\theta}_2 - \overline{\theta}_1)\left[\theta_2\, \hat{\mathcal{X}}^{(k)}(t_1, t_2) + \theta_1\, \hat{\mathcal{X}}^{(k)}(t_2, t_1)\right]. \tag{2.20}$$

Following exactly the same steps as in [11], one gets

$$\boldsymbol{\mathcal{X}}^{(k)}(1, 2) = \frac{1}{\pi} \int_{-2}^{2} \mathrm{d}\lambda \sqrt{1 - \tfrac{1}{4}\lambda^2}\, S_k(\lambda)\, \boldsymbol{g}(\lambda; 1, 2) \tag{2.21}$$

i.e. each component $\mathcal{X}^{(k)}$ and $\hat{\mathcal{X}}^{(k)}$ is the coefficient of the expansion of g and $\hat{g}$ in the polynomials S_k.

One can now show [11] that the $\boldsymbol{\mathcal{X}}^{(k)}$ are obtained from

$$\boldsymbol{\mathcal{X}}^{(k)}(1, 2) = \beta^k\, \boldsymbol{Q}^{k+1}(1, 2) \tag{2.22}$$

† The functions $\boldsymbol{\mathcal{X}}^{(k)}(1, 2)$ are the dynamical analogue of the functions X_k of [11].

or disentagling the superspace notation

$$\mathcal{X}^{(k)}(t_1,t_2) = \frac{1}{\pi}\int_{-2}^{2} \mathrm{d}\lambda \sqrt{1-\tfrac{1}{4}\lambda^2}\, S_k(\lambda)\, g(\lambda;t_1,t_2) = \beta^k\, C^{(k+1)}(t_1,t_2) \tag{2.23}$$

$$\hat{\mathcal{X}}^{(k)}(t_1,t_2) = \frac{1}{\pi}\int_{-2}^{2} \mathrm{d}\lambda \sqrt{1-\tfrac{1}{4}\lambda^2}\, S_k(\lambda)\, \hat{g}(\lambda;t_1,t_2) = \beta^k\, G^{(k+1)}(t_1,t_2)\,. \tag{2.24}$$

We are left with the task of calculating the powers of the superorder parameter Q. Before dealing with this, let us give a compact form for g; using (2.19) and the orthogonality properties of the Chebyshev polynomials we obtain

$$g(\lambda;1,2) = \left[Q\left(\delta - \beta\lambda Q + \beta^2 Q^2\right)^{-1}\right](1,2)\,. \tag{2.25}$$

Products and inverses are as in (2.13) and the identity is defined as

$$\delta(1-2) \equiv (\theta_2-\theta_1)(\overline{\theta}_2-\overline{\theta}_1)\delta(t_2-t_1)\,.$$

The relation (2.25) is valid for all times. It is purely a consequence of the mean-field limit and the (super)symmetries of the problem; we have not yet used at all the dynamical solution.

In the following sections we shall concentrate on the long-time regime (1.3). We shall express the results not in terms of the times, but in terms of the value of the auto-correlation function takes at those times.

3. Powers of Q

We now calculate the powers Q^k for large times. Until explicitly noted, our calculation is not particular to the SK model but only relies on the assumptions made in [6] for the long-time dynamics of mean-field spin-glasses.

For any three large times the auto-correlations satisfy 'triangle relations':

$$C(t_{max},t_{min}) = f\left(C(t_{max},t_{int}),\, C(t_{int},t_{min})\right)\,. \tag{3.1}$$

The function f is an associative composition law.

We also have that

$$G(t_1,t_2) = \frac{\partial F[C(t_1,t_2)]}{\partial t_2} = X_{\mathrm{d}}[C(t_1,t_2)]\,\frac{\partial C(t_1,t_2)}{\partial t_2}\,. \tag{3.2}$$

Equation (3.2) defines $X_{\mathrm{d}}[C]$ and $F[C]$ (the latter up to a constant). It says that the violation of the FDT theorem for the non-equilibrium dynamics of spin-glasses is determined by a function $X_{\mathrm{d}}[C]$ that depends on the times exclusively through $C(t_1,t_2)$.

This scenario has been proposed to analyse the large-time dynamics of the mean-field spin-glass models. The solution of the dynamical problem for a particular model gives explicit expressions for X_{d}, F and f [4, 6].

In the appendices we shall show that the structure (3.1) and (3.2) carries through to Q^k. The reasoning is general and does not depend on the model. The main steps are the following: we first show that $C^{(k)}$ depends on the times only through $C(t_1,t_2)$:

$$C^{(k)}(t_1,t_2) = C^{(k)}[C(t_1,t_2)]\,. \tag{3.3}$$

The triangle relation for $C^{(k)}$ can be read from

$$C\left(C^{(k)}(t_{\max},t_{\min})\right) = f\left(C(C^{(k)}(t_{\max},t_{\mathrm{int}})),\, C(C^{(k)}(t_{\mathrm{int}},t_{\min}))\right) \tag{3.4}$$

i.e. the new triangle relation is isomorphic to the old one.

Relation (3.2) then maps into

$$G^{(k)}(t_1,t_2) = \frac{\partial F^{(k)}[C^{(k)}(t_1,t_2)]}{\partial t_2} = X_{\rm d}^{(k)}[C^{(k)}(t_1,t_2)]\,\frac{\partial C^{(k)}(t_1,t_2)}{\partial t_2}\,. \tag{3.5}$$

In the appendices we also show for the SK model that $X_{\rm d}^{(k)}$ is obtained through

$$X_{\rm d}^{(k)}(C^{(k)}[C]) = X_{\rm d}[C]\,. \tag{3.6}$$

Of particular importance are the values of the correlations $C = a_i^*$ that are 'fixed points' of f

$$f(a_i^*, a_i^*) = a_i^*\,. \tag{3.7}$$

Equation (3.4) implies that fixed points corresponding to C are mapped into fixed points corresponding to $C^{(k)}$.

The fixed points separate the range of auto-correlations in 'discrete scales' [6]. Under very general (model-independent) assumptions, the relation f between two fixed points is ultrametrical

$$f(a_i^*, a_j^*) = \min(a_i^*, a_j^*) \tag{3.8}$$

but not so the relation between values of the auto-correlation that are not fixed points and belong to the same discrete scale.

In general, it turns out that the function $C^{(k)}(C)$, when evaluated in the fixed points a_i^* is related to the ultrametric ansatz in replica space as follows: let Q_{ab} be an ultrametric matrix with elements q_r associated with blocks of sizes X_r. We compute the matrix power $[Q^k]_{ab}$, and consider its elements (say, $q_r^{(k)}$) associated with blocks of size X_r. Then, the functional $q^{(k)}[q]$ coincides with the dynamical functional $C^{(k)}[C]$.

This relationship (at this point purely kinematical) between powers of static and dynamical order parameters holds only for 'fixed point' values of C. The values of q that are not contained as entries of the ultrametric matrix correspond to values of C intermediate between fixed points, i.e. within discrete scales†; for these auto-correlation values there is no replica counterpart within the ultrametric ansatz.

3.1. SK model

For the SK problem in zero magnetic field‡, the solution of the mean-field dynamical equations yields a dense set of fixed points of $f(C,C)$ in the interval $[0, q_{\rm EA}]$, plus an isolated fixed point $C(t,t) = 1$. The value $q_{\rm EA}$ is the Edwards–Anderson parameter, and the interval $(q_{\rm EA}, 1]$ corresponds to the 'FDT ' (discrete) scale. For times associated with auto-correlations in this interval $X_{\rm d}(C) = 1$ and FDT holds. Instead, for large times associated to C in $[0, q_{\rm EA}]$, FDT is modified as in (3.2) by a non-trivial factor $X_{\rm d}[C]$. The function $X_{\rm d}[C]$ is part of the solution to the mean-field equations of motion.

To obtain the explicit form of the powers $\boldsymbol{Q}^k$ it is useful to separate the FDT discrete scale writing $\boldsymbol{Q}(1,2)$ as

$$\boldsymbol{Q}(1,2) = \boldsymbol{Q}_{\rm FDT}(1,2) + \mathcal{Q}(1,2)\,. \tag{3.9}$$

† Let us note, in passing, that the correspondence we have just described is an example of a more general connection between static replica and dynamic SUSY treatments. Indeed this connection holds not only for powers of the order parameters, but for a wide class of functionals $H[Q]$ [13].

‡ This solution has been obtained for T slightly below the critical temperature $T_{\rm c}$. We expect it to hold for all temperatures below $T_{\rm c}$.

The FDT term $Q_{\rm FDT}(1,2)$ has entries that satisfy

$$C_{\rm FDT}(t_1,t_2) = C_{\rm FDT}(t_1-t_2) \tag{3.10}$$

$$G_{\rm FDT}(t_1,t_2) = \frac{\partial C_{\rm FDT}(t_1-t_2)}{\partial t_2}. \tag{3.11}$$

The function $C_{\rm FDT}(\tau)$, $\tau \equiv t_1-t_2$, is a rapidly (with respect to the variation of $\mathcal{Q}$) decreasing function; $C_{\rm FDT}(0) = 1-q_{\rm EA}$ and $C_{\rm FDT}(\infty) = 0$. It is the output of the Sompolinsky–Zippelius dynamics 'within a valley' [3]. Operator powers of $Q_{\rm FDT}$ have entries that verify (3.10) and (3.11) and are relevant in the same time region.

The $\mathcal{Q}$ function varies slowly; $\mathcal{C}(t_1,t_1) = q_{\rm EA}$ and $\mathcal{C}(t_1,t_f) = 0$ if $t_1 \gg t_f$.

In the operator product $Q_{\rm FDT}\,\mathcal{Q}$, the operator $Q_{\rm FDT}$ acts as the identity $\delta(1-2)$ times $1-q_{\rm EA}$ [12].

The separation (3.9) is the dynamic counterpart of the separation of the (static) replica matrix Q_{ab},

$$Q_{ab} = (1-q_{\rm EA})\delta_{ab} + \mathcal{Q}_{ab} \tag{3.12}$$

where $\mathcal{Q}_{ab}$ has $q_{\rm EA}$ in the diagonal.

In order to compute $\mathcal{X}^{(k)}$ we use that, for long times,

$$\begin{aligned} Q^k(1,3) &= \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} \int {\rm d}2\, Q_{\rm FDT}^{k-l}(1,2)\,\mathcal{Q}^l(2,3) \\ &= Q_{\rm FDT}^k(1,3) + ((1-q_{\rm EA})\delta + \mathcal{Q})^k\,(1,3) - ((1-q_{\rm EA})\delta)^k(1,3)\,. \end{aligned} \tag{3.13}$$

This relation allows us to write $\{C^{(k)}, G^{(k)}\}$, the entries of Q^k, in terms of $\{C_{\rm FDT}^{(l)}, G_{\rm FDT}^{(l)}\}$ and $\{\mathcal{C}^{(l)}, \mathcal{G}^{(l)}\}$, the entries of $Q_{\rm FDT}^l$ and $\mathcal{Q}^l$, respectively.

In appendix A we give expressions for the entries $C_{\rm FDT}^{(k)}$ and $G_{\rm FDT}^{(k)}$ of $Q_{\rm FDT}^k$. The explicit form of $C_{\rm FDT}^{(k)}(t_1,t_2)$ has no analogue in the ultrametric ansatz for the replica approach, except for the values at equal times and at times such that $C = q_{\rm EA}$ (i.e. at the limits of the FDT 'discrete' scale). $C_{\rm FDT}^{(k)}$ is also a rapidly decreasing function which falls from $(1-q_{\rm EA})^k$ at equal times to zero at widely separated times. In appendix B we calculate the entries $\mathcal{C}^{(k)}$ and $\mathcal{G}^{(k)}$ of $\mathcal{Q}^k$ for large times.

Using these results we are in a position to express $C^{(k)}$ and $G^{(k)}$ for all ranges of times:

- For large and widely separated times t_1 and t_2 such that $C(t_1,t_2) < q_{\rm EA}$, we compute the sum in (3.13) to get, in terms of $C(t_1,t_2)$,

$$\begin{aligned} C^{(k)}[C] &= \frac{(1-q_{\rm EA}-F[0])^k - (1-q_{\rm EA}-F[C])^k}{X[C]} \\ &\quad - \int_{X[0]}^{X[C]} \frac{{\rm d}x'}{x'^2}\left[\left(1-q_{\rm EA}-F[C(x')]\right)^k - (1-q_{\rm EA}-F[0])^k\right] \end{aligned} \tag{3.14}$$

and

$$G^{(k)}(t_1,t_2) = X_{\rm d}[C]\,\frac{\partial C^{(k)}(C)}{\partial t_2}. \tag{3.15}$$

- For large and close times such that $C > q_{\rm EA}$

$$C^{(k)}(t_1,t_2) = C_{\rm FDT}^{(k)}(t_1-t_2) + C^{(k)}(q_{\rm EA}^-) \tag{3.16}$$

$$G^{(k)}(t_1,t_2) = \frac{\partial C^{(k)}(t_1,t_2)}{\partial t_2} \tag{3.17}$$

with $C^{(k)}(q_{\rm EA}^-)$ from (3.14).

In particular we shall need the result for equal times

$$C^{(k)}(t_1, t_1) = (1 - q_{\text{EA}})^k + C^{(k)}(q_{\text{EA}}^-) . \tag{3.18}$$

4. Expressions for the staggered auto-correlation

Expressions (3.17) and (3.18), together with (3.14) and (3.15), are all that is needed to calculate the staggered auto-correlation and response functions at long times. In order to make contact with the results of [11] we make a change of variables:

$$\Delta[X_{\mathrm{d}}] \equiv F[0] - F[C(X_{\mathrm{d}})]\,\theta(X_M - X_{\mathrm{d}}) \tag{4.1}$$

$$I \equiv F[0] + q_{\text{EA}} \tag{4.2}$$

where $X_M = X_{\mathrm{d}}[q_{\text{EA}}^-]$.

Inverting equation (2.21), using (2.19) and the low-temperature phase result $\beta(1 - I) = 1$, after some algebra we obtain the staggered auto-correlation at long equal times

$$\begin{aligned} g(\lambda) &\equiv \lim_{t\to\infty}\lim_{N\to\infty} g(\lambda, t, t) \\ &= \frac{1}{\beta(2-\lambda)}\left[1 + \int_0^1 \frac{\mathrm{d}X_{\mathrm{d}}}{X_{\mathrm{d}}^2}\left(\frac{(\beta\Delta(X_{\mathrm{d}}))^2}{1 - \lambda(1 + \beta\Delta(X_{\mathrm{d}})) + (1 + \beta\Delta(X_{\mathrm{d}}))^2}\right)\right] . \end{aligned} \tag{4.3}$$

The staggered auto-correlation $g(\lambda, C)$, between two large and widely separated times t_1, t_2 chosen such that $C(t_1, t_2) = C < q_{\text{EA}}$, is given by

$$\begin{aligned} g(\lambda, C) &\equiv \lim_{t_1\to\infty, C(t_1,t_2)=C}\ \lim_{N\to\infty} g(\lambda, t_1, t_2) \\ &= g(\lambda) - \frac{1}{\beta(2-\lambda)}\left[1 + \frac{(1 - \beta(1 - q_{\text{EA}}))^2}{1 - \lambda\beta(1 - q_{\text{EA}}) + \beta^2(1 - q_{\text{EA}})^2}\right. \\ &\quad \left. + \int_{X_{\mathrm{d}}(C)}^{X_{\mathrm{d}}(q_{\text{EA}}^-)} \frac{\mathrm{d}X'_{\mathrm{d}}}{X'^2_{\mathrm{d}}}\left(\frac{(\beta\Delta(X'_{\mathrm{d}}))^2}{1 - \lambda(1 + \beta\Delta(X'_{\mathrm{d}})) + (1 + \beta\Delta(X'_{\mathrm{d}}))^2}\right)\right] . \end{aligned} \tag{4.4}$$

Both these last expressions are valid for the low-temperature phase.

We now note that *for the* SK *model* the functions $X_{\mathrm{d}}(C)$ for the dynamics and the usual function $X(q)$ of the replica treatment coincide at all temperatures. Furthermore, the diagonal values Q_{aa}^k and $C^{(k)}(t_1, t_1)$ also coincide. This also implies the equality of the functions Δ and I.

Hence, we have just proved that for the long and equal times the dynamic staggered spin auto-correlation (4.3) coincides with the static one obtained in [11]. Furthermore, the staggered auto-correlation $g(\lambda, C)$ coincides with the static one† computed with configurations belonging to two equilibrium states with mutual overlap C.

Finally, let us show that, both statically and dynamically, $g(\lambda)$ contains all the information needed to reconstruct $P_0(q)$. To this end we define

$$t(X) \equiv \frac{1 + (1 + \beta\Delta(X))^2}{1 + \beta\Delta(X)} \tag{4.5}$$

and

$$h(\lambda) \equiv \beta\,(2 - \lambda)\,g(\lambda) - 1 . \tag{4.6}$$

† One can extend the equilibrium calculation of [11] to this case by considering the entry of the replica matrices $(1 - \beta\lambda Q + \beta^2 Q^2)^{-1}{}_{ab}$ corresponding to a pair of replicas having mutual overlap $Q_{ab} = C$.

The functions $t(X)$ and $q(X)$ both have a plateau for the same values of $X \in (X_M, 1)$.

Equation (4.3) becomes

$$h(\lambda) = \int_0^{X_M} \frac{\mathrm{d}X}{X^2} \frac{t(X)-2}{t(X)-\lambda} + \left(\frac{1}{X_M} - 1\right) \frac{t(X_M)-2}{t(X_M)-\lambda}. \tag{4.7}$$

Having excluded the plateau in $t(X)$, we can change variables in the integral to obtain

$$h(\lambda) = \int_2^{t_M} \frac{\mu(t')\,\mathrm{d}t'}{t'-\lambda} + \left(\frac{1}{X_M} - 1\right) \frac{t_M-2}{t_M-\lambda} \tag{4.8}$$

where $t_M \equiv t(X_M)$ and $\mu(t) = (t-2)X^{-2}(t)\,\mathrm{d}X/\mathrm{d}t$.

This is an electrostatic problem with positive charges. The determination of $\mu(t), X_M, t_M$ can, in principle, be done in a unique way: the analytical continuation of $h(\lambda)$ from the interval $-2 < \lambda < 2$ yields the 'charge density' $\mu(t)$, the magnitude of the 'discrete charge' and its position t_M.

The knowledge of $\mu(t), X_M, t_M$ then allows us to calculate $X(t)$ as

$$\frac{1}{X(t)} = \int_t^{t_M} \frac{\mu(t')}{t'-2}\,\mathrm{d}t' + \frac{1}{X_M(t_M)}. \tag{4.9}$$

Hence, we have shown that $g(\lambda)$ contains all the information needed to obtain $X(q)$.

5. Numerical simulations

We have performed Monte Carlo simulation of the dynamics of a system with $N = 996$ spins at temperature $T = 0.3$.

We have calculated the distribution of overlaps for two copies of the system relaxing from different initial configurations, at times $t = 600$, 2000 and 10 000 Monte Carlo sweeps. At these times the system is well out of equilibrium, as shown by the form of the overlap distribution. This eventually takes the form of the static $P(q)$ (except for finite-size corrections) at equilibrium, but is only bell-shaped at the times considered (see figure 1).

Figure 2 shows the equal-times staggered auto-correlation times the density of eigenvalues, $\rho(\lambda)g(\lambda, t, t)$, at times $t = 600$, 2000 and 10 000, together with the analytical result for the equilibrium $\rho(\lambda)g(\lambda)$. We notice that the convergence to a curve that coincides with the equilibrium curve is very fast, even in a situation manifestly out of equilibrium (cf figure 1).

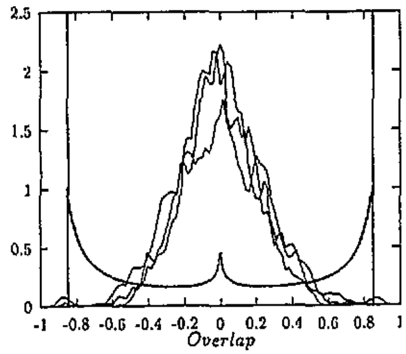

Figure 1. Overlap distributions for $t = 600$, 2000 and 10 000 Monte Carlo sweeps. The broken curve shows the analytical equilibrium $P(q)$.

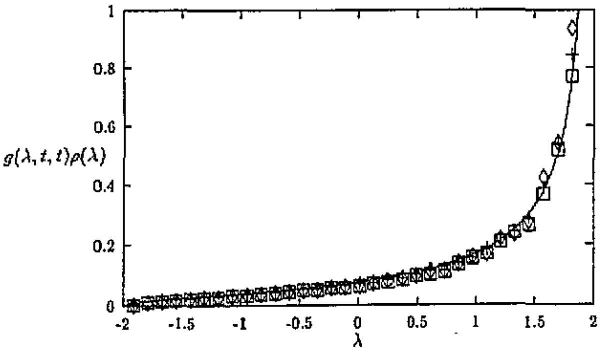

Figure 2. The equal-times staggered auto-correlation distribution $g(\lambda; t, t)$ times the density of eigenvalues $\rho(\lambda)$, for $t = 600$ ($\diamond$), $t = 2000$ (+) and $t = 10\,000$ ($\square$) Monte Carlo sweeps. The full curve shows the analytical result for the statics and for long but finite times.

In particular, the time-dependent energy density is given by

$$e(t) = \int \mathrm{d}\lambda\, \lambda\, \rho(\lambda)\, g(\lambda; t, t)\,. \tag{5.1}$$

Hence, the equivalence $\lim_{t\to\infty} g(\lambda; t, t) = g(\lambda)$ ensures the equivalence of the asymptotic energy and the equilibrium energy, a result that we have also checked numerically.

Let us conclude this section by mentioning that a detailed analysis of the finite-size effects on the off-equilibrium dynamics of the SK model shows that the typical time at which the (saddle-point) mean-field equations do not hold anymore grows algebraically with the system size N [16].

6. Discussion

The partition function of the SK model is dominated by the low-lying states. The out-of-equilibrium dynamics never reaches any of these states: there is never a situation of 'effective' dynamical equilibrium in which the system is trapped forever in one of these states ignoring the rest of the phase space and satisfying FDT and time-translational invariance.

Indeed, as time passes, the evolution of the system slows down more and more but it is never completely trapped. In [6] it was pointed out that the equality $P_{\mathrm{d}}(q) = P(q)$ implied that an infinite SK system has an energy density which goes asymptotically to the equilibrium energy density. Furthermore, the 'width' of the region in which the system has a fast relaxation at long times coincides with the 'size' q_{EA} of the equilibrium states.

This already points to a similarity between the long-time landscape and the (different) region that dominates the partition function. The results in this paper suggest that this similarity is much deeper: consider the relaxation at two large times (t_1, t_2). Because of weak ergodicity breaking [15], given t_1 we can always choose $t_2 > t_1$ such that the auto-correlation $C(t_2, t_1)$ between the configurations $\sigma_i(t_1)$ and $\sigma_i(t_2)$ at those times is any given value C. If we now compute the staggered auto-correlation distribution $g(\lambda, t_1, t_2)$ for those configurations we obtain the same distribution we would have obtained with configurations chosen from two equilibrium states at mutual distance C.

This result is quite surprising, since we know that the system *is not in any equilibrium state* at time t_1 or t_2, however long; it will eventually leave the neighbourhood of $\sigma_i(t_1)$ and $\sigma_i(t_2)$ never to return.

If we now keep the configuration at times t_2 and let the system evolve up to a time t_3 such that again $C(t_2, t_3) = C$ we obtain the same form for the staggered autocorrelation $g(\lambda, t_2, t_3)$. Note however, that because the system slows down, $t_2 - t_1 < t_3 - t_2$ if $C < q_{\mathrm{EA}}$.

The picture that this seems to suggest is that the geometry of phase space seen at different long times is similar in every respect, except that the relevant barriers found at larger times are higher, thus slowing down the system.

We expect that this similarity between the equilibrium and the long-time out-of-equilibrium regions of phase space will hold for models that do not have a 'threshold' level below which the system cannot go. More precisely, we expect this similarity to hold for all models with a continuous set of correlation scales, e.g. the SK model and the model studied in [5, 7], but we do not expect it hold for the p-spin spherical model. The latter model relaxes to a threshold energy density higher than equilibrium, a result that can be intrepreted by the use of the TAP equations [4]. The region of phase space which the system explores at long-times is in the p-spin case completely different from the equilibrium one.

The results we have presented for the SK model do not hold for strictly zero temperature. In that case the system can remain trapped forever in a high-lying metastable state (with barriers not necessarily divergent with N). Furthermore, the result in that case depends on the particular dynamics proposed [17, 18].

Finally, let us remark that the good agreement between the numerical calculation of $g(\lambda, t, t)$ for large t and the static $g(\lambda)$ constitues a rather detailed test of the solution of the out-of-equilibrium dynamics for this model [19, 17].

Appendix A.

In this appendix we give an expression for Q^k_{FDT}. Let us first note that the power of a FDT supersymmetric operator is itself FDT [12]. From equation (2.14) we then have

$$C^{(k)}_{\mathrm{FDT}}(t_1 - t_3) = \left[C^{(k-1)}_{\mathrm{FDT}}(t_1 - t_2)\, C_{\mathrm{FDT}}(t_3 - t_2)\right]_0^{t_3} + \int_{t_3}^{t_1} \mathrm{d}t_2\, C_{\mathrm{FDT}}(t_2 - t_3)\, \frac{\partial C^{(k-1)}_{\mathrm{FDT}}(t_1 - t_2)}{\partial t_2}\,. \tag{A.1}$$

Beacause the time difference $t_1 - t_3$ is in a one-to-one relation with $C_{\mathrm{FDT}}(t_1 - t_3)$, equation (A.1) proves (3.3) for the FDT regime.

The value of $G^{(k)}_{\mathrm{FDT}}(t_1 - t_3)$ is obtained as

$$G^{(k)}_{\mathrm{FDT}}(t_1 - t_3) = \frac{\partial C^{(k)}_{\mathrm{FDT}}(t_1 - t_3)}{\partial t_3}\,. \tag{A.2}$$

This says that $X^{k)}_{\mathrm{d}} = 1$ for C in the FDT regime, and hence is of the form (3.5).

In this paper we only need the value of $C^{(k)}_{\mathrm{FDT}}(t_1, t_1) = C^{(k)}_{\mathrm{FDT}}(0)$. This is easily obtained by putting $t_3 = t_1$ in (A.1). In this way we obtain

$$C^{(k)}_{\mathrm{FDT}}(t_1, t_1) = [C_{\mathrm{FDT}}(t_1, t_1)]^k = (1 - q_{\mathrm{EA}})^k\,. \tag{A.3}$$

From equation (A.1) it is easy to see that $Q^{(k)}_{\mathrm{FDT}}$ is also small for very different times.

Appendix B.

In this appendix we analyse the properties of Q^k for long and widely separated times, for which $Q = \mathcal{Q}$ though $Q^k \neq \mathcal{Q}^k$. We first study $\mathcal{Q}^k$ and then the properties for Q^k will follow from linearity (see equation (3.13)).

We analyse $\mathcal{C}^{(k)}$, $X^{(k)}_{\mathrm{d}}$ and $F^{(k)}$. We first demonstrate by induction that $\mathcal{C}^{(k+1)}$ depends exclusively on $\mathcal{C}$

$$\mathcal{C}^{(k+1)}(t_1, t_2) = \mathcal{C}^{(k+1)}(\mathcal{C}(t_1, t_2))\,. \tag{B.1}$$

Then we show also by induction that the relation between $\mathcal{G}^{(k+1)}$ and $\mathcal{C}^{(k+1)}$ maintains the form (3.5); there exist $F^{(k+1)}$ and $X_{\rm d}^{(k+1)}$ that verify

$$\mathcal{G}^{(k+1)}(t_1,t_2) = \frac{\partial F^{(k+1)}[\mathcal{C}(t_1,t_2)]}{\partial t_2} = X_{\rm d}^{(k+1)}[\mathcal{C}(t_1,t_2)]\,\frac{\partial \mathcal{C}^{(k+1)}(t_1,t_2)}{\partial t_2}\,. \tag{B.2}$$

Finally we explicitly compute $\mathcal{C}^{(k)}$, $X_{\rm d}^{(k)}$ and $F^{(k)}$ in terms of $\mathcal{C}$ for the SK model. The $F^{(k)}$ are defined up to a constant, we fix it by imposing $F^{(k)}(\mathcal{C}(t_1,t_1)) = 0$ (e.g. for $k=1$, $F(q_{\rm EA}) = 0$).

Let us define the (rather badly behaved) 'inverse' of f

$$\mathcal{C}(t_{\max},t_{\min}) = f\left(\mathcal{C}(t_{\max},t_{\rm int}),\mathcal{C}(t_{\rm int},t_{\min})\right) \Longrightarrow \mathcal{C}(t_{\rm int},t_{\min}) = \overline{f}\left(\mathcal{C}(t_{\max},t_{\rm int}),\mathcal{C}(t_{\max},t_{\min})\right)\,. \tag{B.3}$$

The function $\overline{f}$ is written in such a way that its second argument is always smaller than the first one.

We start by assuming

$$\mathcal{C}^{(k)}(t_1,t_2) = \mathcal{C}^{(k)}(\mathcal{C}(t_1,t_2)) \tag{B.4}$$

$$\mathcal{G}^{(k)}(t_1,t_2) = \frac{\partial F^{(k)}[\mathcal{C}^{(k)}(t_1,t_2)]}{\partial t_2} = X_{\rm d}^{(k)}[\mathcal{C}^{(k)}(t_1,t_2)]\,\frac{\partial \mathcal{C}^{(k)}(t_1,t_2)}{\partial t_2} \tag{B.5}$$

with $F^{(k)}[\mathcal{C}^{(k)}(t_1,t_1)] = 0$. Equation (2.14) then reads

$$\begin{aligned}\mathcal{C}^{(k+1)}(t_1,t_3) = &-\mathcal{C}^{(k)}\left[\mathcal{C}(t_1,0)\right]\,F\left[\mathcal{C}(t_3,0)\right] - \int_{\mathcal{C}(t_1,0)}^{\mathcal{C}} {\rm d}y\,\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\,F\left[\overline{f}(\mathcal{C},y)\right]\\ &+\int_{\mathcal{C}(t_1,0)}^{\mathcal{C}} {\rm d}y\,X_{\rm d}^{(k)}\left[\mathcal{C}^{(k)}(y)\right]\,\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\,\overline{f}(\mathcal{C},y)\\ &+\int_{\mathcal{C}}^{q_{\rm EA}} dy\,X_{\rm d}^{(k)}\left[\mathcal{C}^{(k)}(y)\right]\,\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\,\overline{f}(y,\mathcal{C})\end{aligned} \tag{B.6}$$

where $\mathcal{C} \equiv \mathcal{C}(t_1,t_3)$.

It is easy to see that, for all $\overline{f}$ such that $\overline{f}(x,y) \propto y$, $\mathcal{C}(t_1,0) = 0 \Rightarrow \mathcal{C}^{(k)}(t_1,0) = 0$, $\forall k$. In these cases $\mathcal{C}^{(k+1)}$ reads

$$\begin{aligned}\mathcal{C}^{(k+1)}(\mathcal{C}) = &-\int_0^{\mathcal{C}} {\rm d}y\,\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\,F\big[\overline{f}(\mathcal{C},y)\big] + \int_0^{\mathcal{C}} {\rm d}y\,X_{\rm d}^{(k)}\big[\mathcal{C}^{(k)}(y)\big]\,\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\,\overline{f}(\mathcal{C},y)\\ &+\int_{\mathcal{C}}^{q_{\rm EA}} {\rm d}y\,X_{\rm d}^{(k)}\big[\mathcal{C}^{(k)}(y)\big]\,\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\,\overline{f}(y,\mathcal{C})\,.\end{aligned} \tag{B.7}$$

This is a time-reparametrization invariant equality and $\mathcal{C}^{(k)}$ depends on t_1 and t_3 only through $\mathcal{C}$.

We now demonstrate that (B.2) holds $\forall k$. Equation (2.15) implies

$$G^{(k+1)}(t_1,t_3) = \frac{\partial}{\partial t_3}\int_{t_3}^{t_1} {\rm d}t_2\,\frac{\partial F^{(k)}\big[\mathcal{C}^{(k)}(t_1,t_2)\big]}{\partial t_2}\,F\big[\overline{f}\big(\mathcal{C}(t_1,t_2),\mathcal{C}(t_1,t_3)\big)\big]\,. \tag{B.8}$$

Then, we can identify

$$F^{(k+1)}(t_1,t_3) = \int_{t_3}^{t_1} {\rm d}t_2\,\frac{\partial F^{(k)}\big[\mathcal{C}^{(k)}(t_1,t_2)\big]}{\partial t_2}\,F\big[\overline{f}\big(\mathcal{C}(t_1,t_2),\mathcal{C}(t_1,t_3)\big)\big] \tag{B.9}$$

choosing the integration constant to be zero. Now, using $\mathcal{C}^{(k)}(t_1,t_2) = \mathcal{C}^{(k)}(\mathcal{C}(t_1,t_2))$,

$$F^{(k+1)}(t_1,t_3) = \int_{\mathcal{C}}^{q_{\rm EA}} {\rm d}y\,\frac{\partial F^{(k)}\big[\mathcal{C}^{(k)}\big]}{\partial \mathcal{C}^{(k)}}\,\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y}\,F\big[\overline{f}\big(y,\mathcal{C}\big)\big]\,. \tag{B.10}$$

The RHS only depends on $\mathcal{C}$ and then $F^{(k+1)}(t_1, t_3) = F^{(k+1)}[\mathcal{C}^{(k+1)}]$ and $X_{\mathrm{d}}^{(k)}(t_1, t_3) = X_{\mathrm{d}}^{(k+1)}[\mathcal{C}^{(k+1)}]$.

The derivation up to this point is general—it only depends on the assumptions of [6]. For the SK model $\mathcal{C}(t_1, 0) = 0$ for long enough time t_1 and the ultrametric dynamical relation between auto-correlation functions

$$\overline{\mathcal{F}}(x, y) = \min(x, y) = y \tag{B.11}$$

holds [6]. Then $\mathcal{C}^{(k)}(t_1, 0) = 0$, $\forall k$ and

$$\mathcal{C}^{(k+1)}(\mathcal{C}) = -\int_0^{\mathcal{C}} \mathrm{d}y \left(\frac{\partial \mathcal{C}^{(k)}(y)}{\partial y} F[y] + F^{(k)}\left[\mathcal{C}^{(k)}(y)\right] \right). \tag{B.12}$$

We can also solve (B.10) using the ultrametric relation (B.11). We obtain

$$F^{(k)}\left[\mathcal{C}^{(k)}(\mathcal{C})\right] = -(-F[\mathcal{C}])^k \tag{B.13}$$

We now solve (B.12) for $\mathcal{C}^{(k+1)}$. Its derivative WRT $\mathcal{C}$ is

$$\frac{\partial \mathcal{C}^{(k+1)}(\mathcal{C})}{\partial \mathcal{C}} = -\frac{\partial \mathcal{C}^{(k)}(\mathcal{C})}{\partial \mathcal{C}} F[\mathcal{C}] - F^{(k-1)}\left[\mathcal{C}^{(k-1)}\right]. \tag{B.14}$$

Using (B.13) we get the recursive equation

$$w^{(k+1)} = w^{(k)} + 1 \tag{B.15}$$

with $w^{(k)} \equiv (-F[\mathcal{C}])^{k-1}\, \partial \mathcal{C}^{(k)}(\mathcal{C})/\partial \mathcal{C}$. The solution is

$$\frac{\partial \mathcal{C}^{(k)}(\mathcal{C})}{\partial \mathcal{C}} = k\,(-F[\mathcal{C}])^{k-1} \tag{B.16}$$

$$\mathcal{C}^{(k)}(\mathcal{C}) = -\int_0^{\mathcal{C}} \mathrm{d}y \, \frac{\partial (-F[y])^k}{\partial y} \frac{1}{X_{\mathrm{d}}[y]} . \tag{B.17}$$

We now obtain $X_{\mathrm{d}}^{(k)}[\mathcal{C}^{(k)}(\mathcal{C})]$ in terms of $X_{\mathrm{d}}[\mathcal{C}]$. Differentiating (B.13) WRT $\mathcal{C}$

$$X_{\mathrm{d}}^{(k)}[\mathcal{C}^{(k)}(\mathcal{C})] \, \frac{\partial \mathcal{C}^{(k)}(\mathcal{C})}{\partial \mathcal{C}} = k\,(-F[\mathcal{C}])^{k-1}\, X_{\mathrm{d}}[\mathcal{C}] \tag{B.18}$$

and inserting the result in (B.16) we obtain

$$X_{\mathrm{d}}^{(k)}[\mathcal{C}^{(k)}(\mathcal{C})] = X_{\mathrm{d}}[\mathcal{C}] . \tag{B.19}$$

We have obtained these results $\mathcal{C}^{(k)} = \mathcal{C}^{(k)}(\mathcal{C})$ and (B.19) for the powers $\mathcal{Q}^k$. Since these hold for every power k and $\mathcal{C}^{(k)}$ is a linear combination of these powers (see equation (3.13)) we have proved equation (3.3) and (3.6).

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